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A Tractable Approach for One-Bit Compressed Sensing on Manifolds

Compressed Sensing

Recover unknown signal $\mathbf{x} \in \mathbb{R}^D$ from $m \ll D$ measurements

$$\mathbf{y} = \mathbf{A}\mathbf{x} \tag{1}$$

under following assumptions:

- $\mathbf{x}$ is s-sparse, i.e., at most s entries are non-zero
- measurement matrix $\mathbf{A} \in \mathbb{R}^{m \times D}$ is known
- $\mathbf{y} \in \mathbb{R}^m$ with $y_l = \langle \mathbf{a}_l, \mathbf{x} \rangle$, $l = 1, \ldots, m$ is given.

Recovery: Sparsity of $\mathbf{x}$ allows recovery by efficient algorithms with

$$m \geq Cs \log \left(\frac{D}{s} \right)$$

measurements where $C > 0$ is an absolute constant.

Problem Formulation

Recover unknown signal $\mathbf{x} \in \mathbb{R}^D$ from $m \ll D$ one-bit measurements

$$\mathbf{y} = \text{sign}(\mathbf{A}\mathbf{x}) \tag{2}$$

under following assumptions:

- $\mathbf{x} \in \mathscr{M}$, where $\mathscr{M}$ is a low-dimensional manifold of intrinsic dimension $d \ll D$ which lies on the unit sphere $\mathbb{S}^{D-1}$
- measurement matrix $\mathbf{A} \in \mathbb{R}^{m \times D}$ with iid Gaussian entries
- $\mathbf{y} \in \{-1, 1\}^m$ is given

Observation: One-bit measurements of type (2) tessellate $\mathbb{S}^{D-1}$ into a collection of distinguishable cells (see Figure 1).

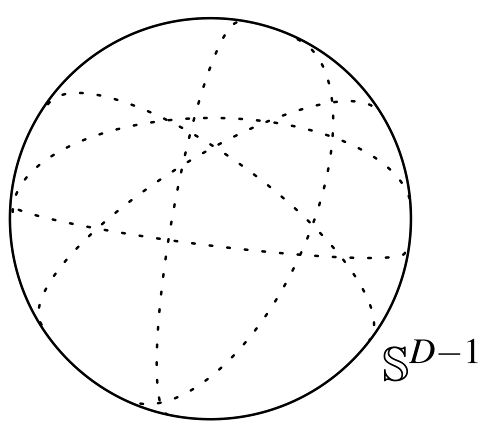

Figure 1: Tessellation of the sphere.

Locally linear approximation of $\mathscr{M}$ via ***G**eometrical **M**ulti-**R**esolution **A**nalysis*:

Definition GMRA, [1]

Let $J \in \mathbb{N}$ and $K_0, K_1, \ldots, K_J \in \mathbb{N}$. For each $j \in [J] := \{0, \ldots, J\}$ we assume to have sets $\mathscr{C}_j \subset \mathbb{R}^D$ of centers and

$$\mathscr{P}_j = \{\mathbb{P}_{j,k} : \mathbb{R}^n \to \mathbb{R}^n | k \in [K_j]\}$$

of corresponding affine projectors which approximate $\mathscr{M}$ at scale j. These form a GMRA for $\mathscr{M}$ if several assumptions (see [1]) are met.

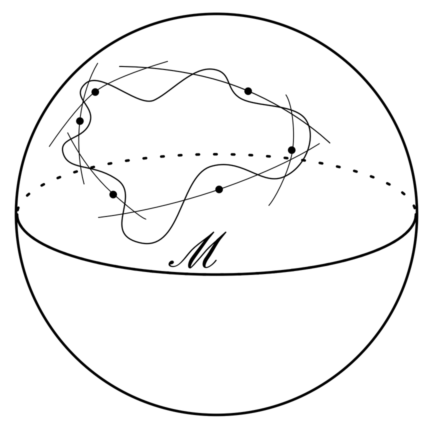

Figure 2: Submanifold $\mathscr{M}$ of $\mathbb{S}^{D-1}$ and one level of GMRA.

Observation: GMRA represents $\mathscr{M}$ as combination of:

- anchor points (the centers $\mathbf{c}_{j,k} \in \mathscr{C}_j$)
- low dimensional affine spaces ($P_{j,k} = \mathbb{P}_{j,k}(\mathbb{R}^n)$)

The levels j control accuracy of the approximation.

Aim: Reconstruct an approximation $\hat{\mathbf{x}}$ of $\mathbf{x}$ from the one-bit measurements $\mathbf{y}$ exploiting the low-dimensional manifold structure of $\mathscr{M}$ (see Figure 3).

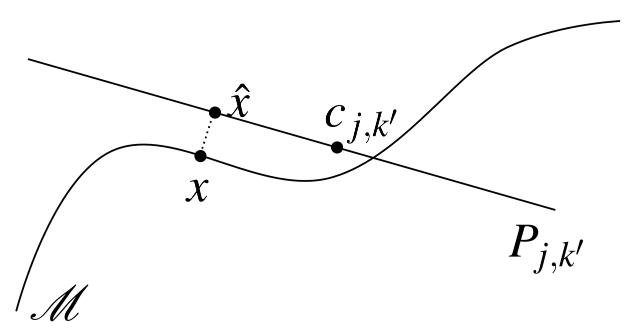

Figure 3: Approximation of x.

Reconstruction Algorithm

Algorithm: One-bit Manifold Sensing (OMS)

I. Identify center $\mathbf{c}_{i,k'}$ close to $\mathbf{x}$ via

$$\mathbf{c}_{j,k'} \in \arg\min_{\mathbf{c}_{j,k} \in \mathscr{C}_j} d_H(\text{sign}(\mathbf{A}\mathbf{c}_{j,k}), \mathbf{y})$$

where d_H is the Hamming distance, i.e., $d_H(\mathbf{z}, \mathbf{z}') := |\{l : z_l \neq z'_l\}|$.

II.

- *If $d_H(\text{sign}(\mathbf{A}\mathbf{c}_{j,k'}), \mathbf{y}) = 0$, directly choose $\hat{\mathbf{x}} = \mathbf{c}_{j,k'}$.*
- *If not, recover the projection of $\mathbf{x}$ onto $P_{j,k'}$, i.e., $\mathbb{P}_{j,k'}(\mathbf{x})$ by*

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{z} \in \mathbb{R}^D} \sum_{l=1}^{m} (-y_l) \langle \mathbf{a}_l, \mathbf{z} \rangle, \tag{3}$$

$$\text{subject to } \mathbf{z} \in \text{conv}\left(\mathbb{P}_{\mathbb{S}}\left(P_{j,k'} \cap \mathscr{B}(\mathbf{0}, 2)\right)\right)$$

This reconstruction strategy combines

I. compressed sensing for signals on general manifolds (Iwen and Maggioni [2])

II. noisy one-bit compressed sensing (Plan and Vershynin [3]).

Main Result

Notation: Denote by $w(\mathscr{M})$ the *Gaussian mean width*

$$w(\mathscr{M}) = \mathbb{E} \sup_{\mathbf{z} \in \mathscr{M} - \mathscr{M}} |\langle \mathbf{g}, \mathbf{z} \rangle|, \quad \mathbf{g} \sim \mathscr{N}(\mathbf{0}, \mathbf{I}_n),$$

which reflects the manifold's complexity.

Theorem

There exist constants $E, E', c > 0$ depending on the GMRA quality such that the following holds. Let $\varepsilon \in (0, 1/16]$ and assume $J \geq \lceil 10 \log(1/\sqrt{\varepsilon}) \rceil$. If

$$m \geq E\varepsilon^{-7} \max \left\{ w(\mathscr{M}), \sqrt{d \log(1/\sqrt{\varepsilon})} \right\}^2$$

Then, with probability at least $1 - 12\exp(-c\varepsilon^2 m)$ for all $\mathbf{x} \in \mathscr{M}$ the approximation $\hat{\mathbf{x}}$ fulfills

$$\|\mathbf{x} - \hat{\mathbf{x}}\|_2^2 \leq E'\varepsilon.$$

If the GMRA is created from random samples of $\mathscr{M}$ the same result holds true with slightly changed constants and probability.

Numerical Simulation

Based on GMRA code provided by Mauro Maggioni, with following parameters:

- 20000 data points sampled from the 2-dimensional sphere $\mathscr{M}$ embedded in $\mathbb{S}^{20-1}$
- fixed GMRA computed up to $J = 10$ refinement levels
- recovery of 100 randomly chosen x lying on $\mathscr{M}$

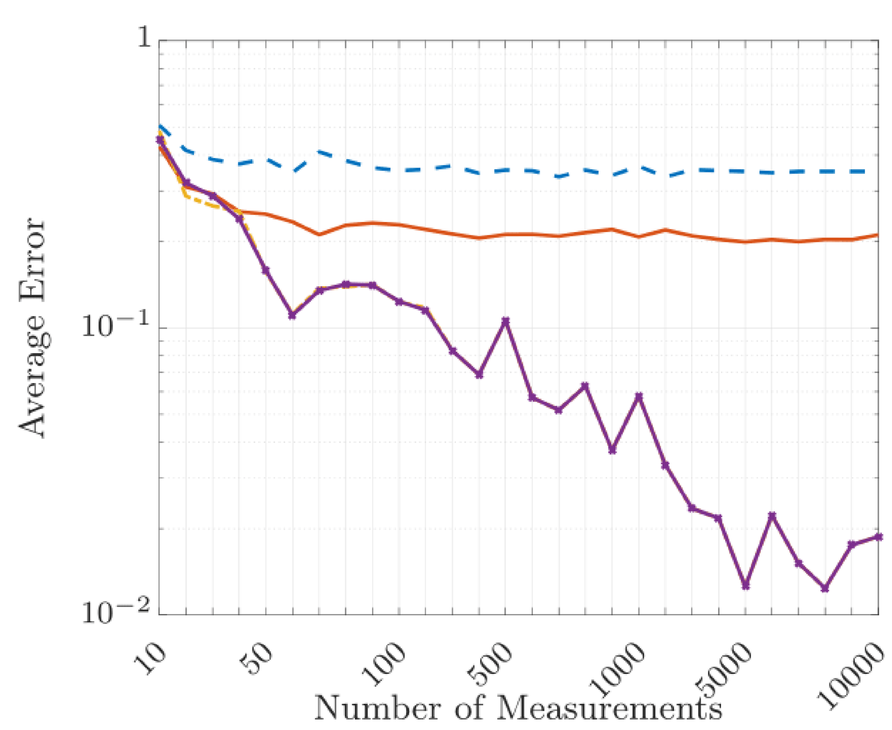

Figure 4: Comparison of the following: Approximation by step I of OMS when using tree structure (dashed, blue) and when comparing all centers (solid, red); approximation by step I+II of OMS when using tree structure (dashed with points, yellow) and when comparing all centers (solid with points, purple).

References

[1] W. Allard, G. Chen, and M. Maggioni, "Multi-scale geometric methods for data sets ii: Geometric multi-resolution analysis," *Applied and Computational Harmonic Analysis*, vol. 32, no. 3, pp. 435–462, 2012, iSSN 1063-5203.

[2] M. A. Iwen and M. Maggioni, "Approximation of points on low-dimensional manifolds via random linear projections," *Information and Inference: A Journal of the IMA*, vol. 2, no. 1, p. 1, 2013.

[3] Y. Plan and R. Vershynin, "Robust 1-bit compressed sensing and sparse logistic regression: A convex programming approach," *CoRR*, vol. abs/1202.1212, 2012. [Online]. Available: http://arxiv.org/abs/1202.1212

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