

Learning interpolation by replication

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Introduction

Interpolation of rates has been typically been treated as a curve smoothness problem:

Inputs:

- Maturities of traded bonds: $t_j = \{t_1, t_2, \dots, t_N\}$
- Rates (yields) of traded (zero coupon) bonds: $r_j = \{r_1, r_2, \dots, r_N\}$

Different methods can be used:

- Linear interpolation:
 - $r_T = r_1 + \left(\frac{T-t_1}{t_2-t_1}\right)(r_2 - r_1)$
- Log-linear interpolation:
 - $T \ln(1 + r_T) = t_1 \ln(1 + r_1) + \left(\frac{T-t_1}{t_2-t_1}\right)(t_2 \ln(1 + r_2) - t_1 \ln(1 + r_1))$
- Splines:
 - Cubic
 - Forward monotone convex

Characteristics of the methods:

- Locality: dependence only on rates r_1, r_2 where t_1, t_2 are:
 - $t_1 = \max_{t_j \leq T} t_j$
 - $t_2 = \min_{t_j \geq T} t_j$
- Smoothness of the forward
- Are rates bounded? Tradeoff between smoothness of forwards and spot rates
- Arbitrageable? Avoid negative forwards

Replication

But the problem is not only finding the best rate and looking a final P&L like:

$$V_0 = (1 + r_0)^{-T} \rightarrow 1 - V_0 \times Rf_{0,T}$$

In fact the interpolation problem is a replication problem, quite similar to the pricing of an option:

One buys a bond for a broken date at a price given by an interpolation method and hedges the interest rate risk by selling traded bonds in a proportion determined by the same interpolation method.

So the final P&L of the position depends not only on the realized interest rates as above, but also on the hedging portfolio:

$$\Pi_T = 1 - V_0 \times Rf_{0,T} + \sum_j \sum_{t=1}^T (w_{j,t-1} dP_{j,t} Rf_{t,T})$$

Where:

- $w_{j,t-1}$ is the weight of each bond j used in the hedging portfolio at time $t-1$
- $dP_{j,t} = P_{j,t} - P_{j,t-1} Rf_{t-1,t}$ is the difference between the price of bond j at time t and the price of the same bond at time $t-1$ increased by the funding cost
- $Rf_{t,T}$ is the factor (>1) corresponding to the accrual of overnight funding costs between t and T

So the method that determines the rate used to calculate V_t is also used to calculate the weights w_j ; but where is the sensitivity to the covariance of all the future price changes of the hedging instruments?

What if they are positively correlated? Negatively correlated? What if volatilities and correlations change with time?

An ideal interpolation method would take into account all of the (historical + market implied) covariance structure (including time decays) and find the optimal hedging policy as the policy that minimizes the variance of the P&L of the portfolio.

It is important to stress this point: Two interpolation methods could give you the same rate at the start of the trade, but they'll provide you with different hedging ratios for the same traded bonds.

Over time, the P&L of each of the hedged portfolios will be different.

Which is right? No method is right on itself, but one can test the performance of the methods on different dynamics for the interest rates:

- Establish the (stochastic) model for the interest rates
- Simulate the co-evolution of the overnight rates and the market rates
- Calculate the interpolated rates and hedging portfolios as each path
- Measure and compare the distribution of the P&Ls at different maturities
- Chances are that you'll find a large bias and a large variance

The best method for each dynamic is the one that minimizes bias and variance of the hedged portfolio

Learning

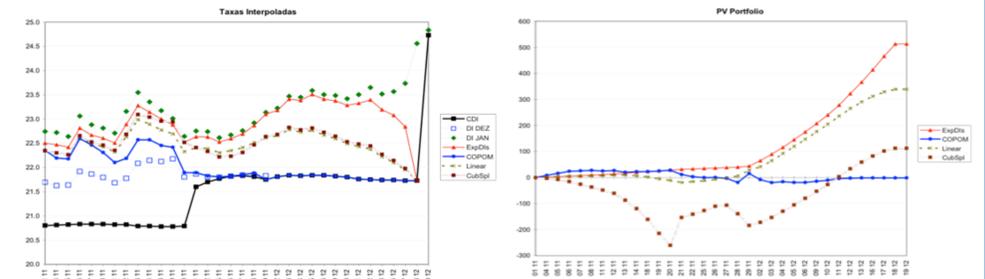
What can we do with past data?

We can learn when bonds cease to be informative. This happens when the P&L of holding this bond in your hedging portfolio only adds noise to it.

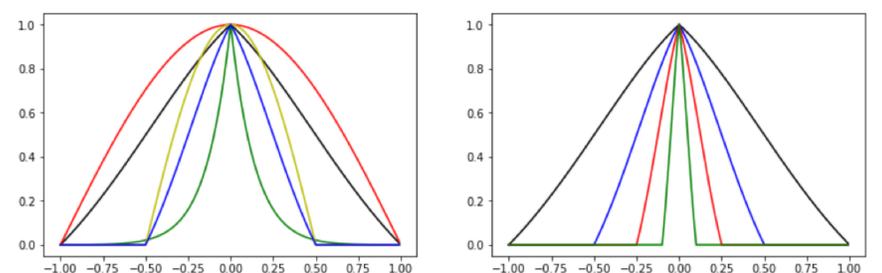
When will this happen?

In Brazil, the interest rate curve is traded through futures that replicate Overnight Interest Rate Swaps; the overnight interest rate (the CDI) is pegged to the overnight interest rate on reverse repos traded with the Central Bank (SELIC) which tracks the target rate determined periodically by the COPOM (Monetary Policy Committee).

So, once there's a meeting the overnight rate is going to be the same for 1.5 months. This means that, if I have an IR Swap maturing 40 days after a COPOM meeting, the contract maturing in 2 months will reflect only uncertainty about the next meeting, and should not be used to hedge this swap.



Generalizing, we can define the weight of each bond at each trading date using a simple function of the distance of the maturity of the target to the maturity of the bond, scaled by the distance between the bond and the closest reference bond on the other "side" of the target:



On the left we have two parameters, on the right only one parameter (for each side).

Because we would use the same parameter (on each side) for all the maturities at a particular trade date, we can be able to find the best value of the parameter given its influence on the P&L of several maturities by minimizing the bias and variance of several portfolios (corresponding to several maturities with the same starting date).

By capturing the dates where weights start retracting (black to blue to red to green on the right chart), we could infer the dates of the COPOM meetings and when a fixing happens.

This can be extrapolated to test interpolation within the market:

How informative are all the points?

- Choose a market point as a target, hedging it with others and your previously tested interpolation method
- Test the P&L; if there's no significant bias or variance, the point is redundant and in this way you can reduce the dimensionality of the curve
- Autocorrelation of Central Bank decisions means interpolation needs past information
 - Next decisions are conditioned on past decisions
 - Smoothness of (discrete) changes in target rate
- Movements of the curve are dependent on the existing shape of the curve
 - Use the previously suggested method to reduce dimensionality of the curve to classify "shapes"

Modeling hidden variables with conditioning on transition matrices

References

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- "Mathematical Techniques in Finance – Tools for Incomplete Markets" (2ed), Černý

Contact

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Link to paper and Jupyter notebook on the page soon