Kendall’s tau: a measure of conditional dependence

Conditional Kendall’s tau between $X_1$ and $X_2$ given $Z = z$:

\[ \tau_{1,2|Z=z} = \tau(C) := \frac{1}{n} \sum_{i=1}^{n} \left( \frac{(X_{1i} - X_{2i})}{\tau(X_1, X_2)} (X_{2i} - X_{1i}) > 0 \right) \]

where $X_{1i}, X_{2i}, Z_{ij}$ are i.i.d. copies of a random vector $(X, Z) \in \mathbb{R}^{1+p}$. 

A classification point-of-view

For a sample $(X_{1i}, X_{2i}, Z_{ij})$, $i = 1, \ldots, n$,

\[ \hat{\tau}_{1,2|Z=z} := \frac{1}{n} \sum_{i=1}^{n} w_i(z) w_{ij}(z) \times \left( \frac{1}{n} (X_{1i} - X_{2i})(X_{2i} - X_{1i}) > 0 \right) \]

where $w_i(z) := K_h(Z_i - z)$ for a kernel $K$ on $\mathbb{R}^p$ and a bandwidth $h > 0$.

A discordant pair

Figure: The “simplifying assumption”: $Z$ has an influence on the conditional margins $X_1$ and $X_2$, but not on the conditional dependence between them.

Goals

- Modeling the influence of $Z$ on the dependence between $X_1$ and $X_2$
- With a conditional dependence parameter that always exists
- And is invariant by changes of margins and scales.

A kernel-based estimator

Figure: Estimated conditional Kendall’s tau between French and German daily stock returns given the intraday volatility $\sigma = \frac{\text{High} - \text{Low}}{\text{Close}}$ during the European debt crisis (2009-2012) (left) and during the following period (2012-2019).

References