Hedging option contracts with locally weighted regression, functional data analysis, and Markov chain Monte Carlo techniques

Jae Yun JUN KIM and Yves RAKOTONDRATSIMBA **Contact: jae-yun.jun-kim@ece.fr** **Applied Mathematics, ECE Paris**

Context

The option hedging is often realized by replicating the changes of option prices with the Delta-Gamma Approximation (DGA). This approximation works well when the changes of the underlying asset price remain small. However, when this change becomes large, the option price estimated by the DGA can be significantly different from the actual value. Hence, before the return of the underlying asset price becomes large, rebalancing operations are demanded. The frequency of re-balancing may be high when the rate of the change of the underlying asset price is significant. Nonetheless, frequent re-balancing positions may be unattainable, as there are associated transaction costs. Hence, there is a trade-off between costs incurring from frequent hedging operations and losses resulting from the inaccurate performance of the DGA.

Objective

- **(LWR)**: the significant error gap that exists between the actual call option price (in this work, estimated by the Black-Scholes model) and the one that is estimated by the DGA is improved using the locally weighted regression (LWR), requiring only the first-order term. This aspect is positive because only one instrument is required for hedging.
- **(FDA)**: the change of the underlying asset prices for future time horizons is forecasted using the functional data analysis (FDA) approach. This forecast is needed to estimate the call option price using both the Black-Scholes model and the LWR approach.
- **(MCMC)**: the implied volatility of an option contract is forecasted using the Markov chain Monte Carlo (MCMC) techniques. This forecast is needed to estimate option prices more accurately.

In summary, our contribution is the proposition of a framework to hedge options by forecasting 1) the underlying stock-price return (with FDA), 2) the corresponding the implied volatility (with MCMC), and 3) the corresponding option price (with LWR).

Literature review

The Black-Scholes price

$$C_t \triangleq C^{(bs)}(S_t, T-t, \sigma_t, r_t, K) = S_t\Phi(d_+) - K\exp[-r_t(T-t)]\Phi(d_-),$$

where S_t is the **underlying asset price**, T is the **maturity time**, σ_t is the **implied volatility of the price** C_t, $r_t > 0$ is the **prevailing interest rate** (free of credit risk), K is the **strike price**, Φ is the **cumulative distribution function** of the standard Gaussian law, and $d_\pm$ are defined as

$$d_\pm \triangleq \frac{1}{\sigma_t\sqrt{T-t}}\left(\ln\left(\frac{S_t}{K}\right) + \left(r_t \pm \frac{1}{2}\sigma_t^2\right)(T-t)\right).$$

The following simplifications are often made in the literature

$$C_t - C_0 = C^{(bs)}(S_t, T-t, \sigma_0, r_0, K) - C^{(bs)}(S_0, T, \sigma_0, r_0, K),$$

Delta Gamma Approximation (DGA)

$$C_t - C_0 \approx \Delta_0 S_0 v_t + \frac{1}{2}\Gamma_0 (S_0 v_t)^2 = \Delta_0(S_t - S_0) + \frac{1}{2}\Gamma_0(S_t - S_0)^2,$$

where $v_t \triangleq \dfrac{S_t - S_0}{S_0}$

is the relative change (or linear return) of the underlying asset value during the time period $(0,t)$, and Δ_0 and Γ_0 are defined as

$$\Delta_0 \triangleq \frac{\partial}{\partial S}\left(C^{(bs)}(S, T, \sigma_0, r_0, K)\right)\Big|_{S=S_0} = \Phi(d_+(0)),$$

$$\Gamma_0 \triangleq \frac{\partial^2}{\partial S^2}\left(C^{(bs)}(S, T, \sigma_0, r_0, K)\right)\Big|_{S=S_0} = \frac{\rho(d_+(0))}{\sigma_0 S_0\sqrt{T}},$$

where ρ is the probability density function of the standard Gaussian law.

Extended Delta Gamma approximation (EDGA)

$$\begin{aligned} C_t - C_0 &= C^{(bs)}(S_t, T-t) - C^{(bs)}(S_0, T) \\ &= C^{(bs)}(S_t, T-t) - C^{(bs)}(S_t^*, T-t) + C^{(bs)}(S_t^*, T-t) - C^{(bs)}(S_0, T) \\ &= \alpha_t + \Delta_t^*(S_t - S_t^*) + \frac{1}{2}\Gamma_t^*(S_t - S_t^*)^2, \end{aligned}$$

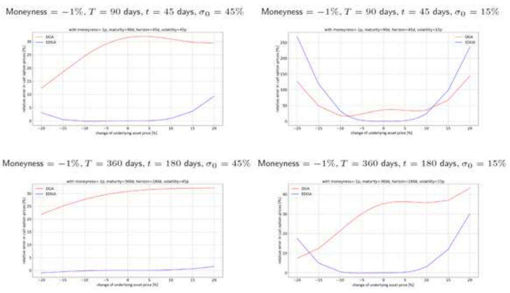
where $\alpha_t = C^{(bs)}(S_t^*, T-t) - C^{(bs)}(S_0, T)$,

$$\Delta_t^* \triangleq \frac{\partial}{\partial S}\left(C^{(bs)}(S, T-t, \sigma_0, r_0, K)\right)\Big|_{S=S_t^*} = \Phi(d_+^*(t)),$$

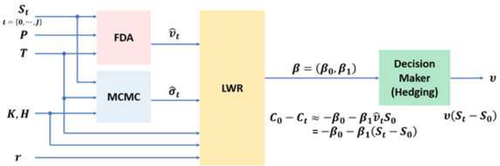
$$\Gamma_t^* \triangleq \frac{\partial^2}{\partial S^2}\left(C^{(bs)}(S, T-t, \sigma_0, r_0, K)\right)\Big|_{S=S_t^*} = \frac{\rho(d_+^*(t))}{\sigma_0 S_t^*\sqrt{T-t}},$$

$$d_+^* \triangleq \frac{1}{\sigma_0\sqrt{T-t}}\left(\ln\left(\frac{S_t^*}{K}\right) + \left(r_0 \pm \frac{1}{2}\sigma_0^2\right)(T-t)\right).$$

Performance comparison: DGA and EDGA



Methodology



S_t: The underlying stock price at time t
P: Periodicity for the FDA
T: Maturity time (Expiration time)
K: Strike
H: Horizon time
r: risk-free interest rate
$\hat{v}_t$: Predicted underlying stock-price return for time t
$\hat{\sigma}_t$: Predicted implied volatility for time t
β: Parameters of the LWR
C_t: Call option price at time t
v: Number of shares to be bought at time 0 for hedging
FDA: Functional Data Analysis
MCMC: Markov chain Monte Carlo
LWR: Locally Weighted Regression

Locally weighted regression (Review)

$$\min_{\hat\beta} \frac{1}{2}\sum_{p=1}^{P} w_p\left(C^{(bs)} - X\hat\beta\right)^2$$

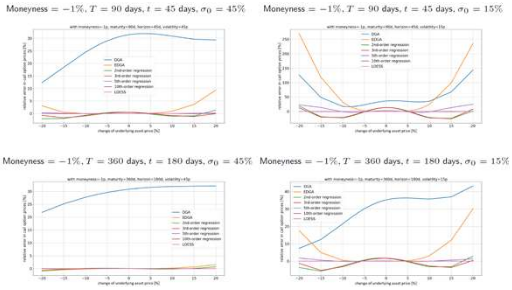
where

$$X = \begin{bmatrix} 1 & v_1 & v_1^2 & \cdots & v_1^N \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & v_P & v_P^2 & \cdots & v_P^N \end{bmatrix} \in \mathbb{R}^{P\times(N+1)},\ C^{(bs)} \in \mathbb{R}^{P\times 1},\ \hat\beta \in \mathbb{R}^{(N+1)\times 1},$$

For each entry v_j, we determine the coefficients for the locally weighted regression

$$\hat\beta = (X^TWX)^{-1}X^TWC^{(bs)}, \qquad W \in \mathbb{R}^{P\times P}$$

Performance comparison: DGA, EDGA, OLS and LWR



Results obtained using the LOESS approach with $N=1$

Forecasting v_t using the functional data analysis (FDA)

In FDA, we model a scalar response via a functional linear model

$$y_i = \beta_0 + \int_0^T X_i(s)\beta(s)ds + \epsilon_i, \quad \text{for } i = 1, \cdots, N.$$

We expand the β in some basis

$$\beta(s) = \sum_{k=1}^{K} b_k\theta_k(s) \quad \Leftrightarrow \quad \beta(s) = \theta'(s)b.$$

We also expand the covariate functions in some basis

$$X_i(s) = \sum_{\ell=1}^{L} c_{i\ell}\psi_\ell(s) \quad \Leftrightarrow \quad X(s) = C\psi(s).$$

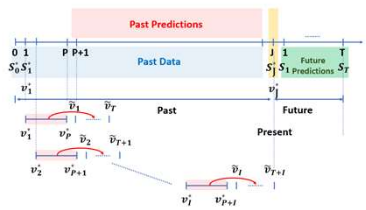
If we now let $J = \int \psi(s)\theta'(s)$, then our model can be expressed as

$$y = \beta_0 + CJb.$$

If we let $Z = \begin{bmatrix}1 & CJ\end{bmatrix}$ and $\xi = \begin{bmatrix}\beta_0 & b'\end{bmatrix}$, then our model is

$$y = Z\xi, \qquad \text{where} \quad \hat\xi = \left(Z'Z\right)^{-1}Z'y.$$

FDA: data pre-processing



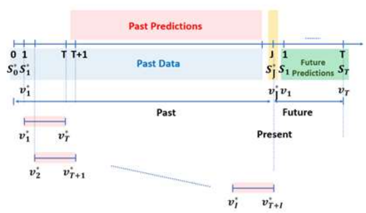
Gibbs sampling for estimating the implied volatility

- Gibbs sampling is a Markov chain Monte Carlo algorithm for obtaining a sequence of observations which are approximated from a multivariate probability distribution when direct sampling is difficult.
- This sequence can be used to:
  - approximate the joint distribution,
  - approximate the marginal distribution of one or a subset of variables,
  - compute an integral such as the expected value of some variables

Algorithm 1: Gibbs sampling

Given some evidence (data): X, y;
Initialize $\Theta^{(0)} = \left(\theta_0^{(0)}, \cdots, \theta_N^{(0)}\right)$;
while $i \leq I$ **do**
  For each component of $\Theta^{(i)} = \left(\theta_0^{(i)}, \cdots, \theta_N^{(i)}\right)$,
  $\theta_j^{(i)} \sim P\left(\theta_j^{(i)} \middle| \tilde\Theta, x^{(i)}, y^{(i)}\right) \propto P\left(y^{(i)} \middle| \theta_j^{(i)}, x^{(i)}, \tilde\Theta\right) P\left(\theta_j^{(i)}\right)$;
  where $\tilde\Theta = \left(\theta_0^{(i)}, \cdots, \theta_{j-1}^{(i)}, \theta_{j+1}^{(i-1)}, \cdots, \theta_N^{(i-1)}\right)$.
Result: $\Theta = \left(\Theta^{(0)}, \cdots, \Theta^{(I)}\right)$

Gibbs sampling: data pre-processing



Results

Assumptions

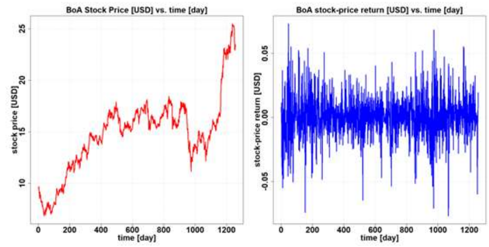
- In this work, we only deal with the European call option on the contract writer (seller) side.
- The option contract writer (seller) can hedge her investment by investing on the underlying asset at any time instant prior to the maturity time.
- In this work, we only deal with a single-period hedging.
- The bid and the ask prices are equal (i.e., the bid-ask spread is zero.)
- The risk-free interest rate is known and constant (2.5%).
- $T = 30$ days; $\log(\text{moneyness}) = 0.01$.
- Implied volatility values generated using the Malz model:

$$\hat\sigma_t = \alpha_0 + \alpha_1\frac{1}{\sqrt{T-t}}\log\left(\frac{S_t}{K}\right) + \alpha_2\left(\frac{1}{\sqrt{T-t}}\log\left(\frac{S_t}{K}\right)\right)^2,$$

  where the parameters α_0, α_1 and α_0 follow some uniform distributions.
- Call option prices are generated using the Black-Scholes model.

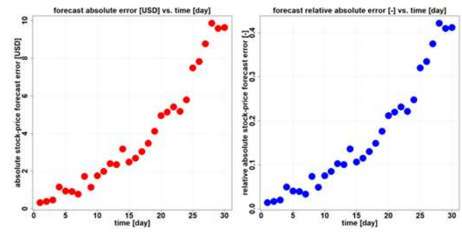
Data for evaluating the forecast performance

Stock data source: Yahoo! Finance
Considered asset: Bank of America (in US Dollar)
Period: March 29, 2012 - March 28, 2017 (1257 samples)

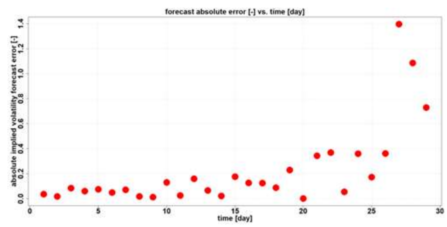


FDA: Error between actual and forecasted stock price

- R Package: FDA / FDA.USC
- FPCA
- 11 basis for both the parameters and the variables.
- Periodicity: 132 days



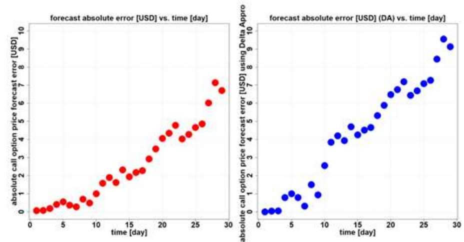
MCMC: Error b/w actual and estimated implied volatility



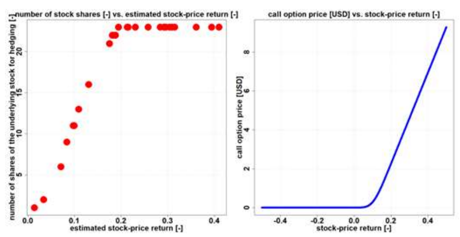
LWR: Error b/w actual and estimated call option prices

- R package: LOESS
- First-order LWR → hedge by reinvesting on a single instrument

Below we compare the results of our method and of the DA approach.



LWR: Number of shares of the underlying stock



Future work

Extend the present work by searching for multi-period actions over various options:

$$\pi^* = \underset{\mathbf{a}=\{a_0,\cdots,a_{T-1}\}}{\arg\max}\ \mathbb{E}_{\mathbb{P}}\left[C_0 - C_T - \sum_{t=0}^{T-1}\gamma^t R(s_t, a_t) - F(s_t, a_t)\,|\mathcal{F}\right]$$